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Non-Maximally Entangled Controlled Teleportation Using Four Particles Cluster States

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Abstract A new scheme for controlled teleportation with the help of a four-qubit cluster state is proposed. In this scheme, a four-particle cluster state is shared by a sender, a controller and a receiver. The sender first performs a Bell-basis measurement on the qubits at hand, and the controller performs measurements under a non-maximally entangled Bell-basis after he knows the sender's measurement result. Then the receiver introduces an auxiliary qubit and performs some appropriate unitary transformations on his qubits. Quantum teleportation is realized after the receiver performs a local measurement on the auxiliary qubit and an appropriate unitary transformation on his qubit.

Keywords Controlled teleportation · Cluster state · Non-maximally entangled state

1 Introduction

Since Bennett et al. [1] presented the protocol of quantum teleportation in 1993, one of the most amazing features of quantum mechanics, quantum teleportation has been studied by many groups [2–7]. The first controlled teleportation protocol was proposed in 1998 by using a three-qubit Greenberger-Horne-Zeilinger (GHZ) state [8]. The basic idea of controlled teleportation is to transport an unknown quantum state with a controller. It has been under extensive investigation [9–26]. Li et al. proposed a scheme to teleport an unknown single-qubit state with a multi-particle joint measurement [10]. Yang et al. presented a multiparty controlled teleportation protocol to teleport multi-qubit quantum information [11]. Deng et al. introduced a symmetric multiparty controlled teleportation scheme for an arbitrary two-particle entanglement state [12]. Man et al. presented an explicit genuine $(2N+1)$-qubit

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entangled state motivated by the idea of controlled teleportation [13]. Zhou et al. [14] presented a theoretical scheme for controlled quantum teleportation, using the entanglement properties of GHZ state. Yan and Wang [15] proposed a protocol for the controlled teleportation of an unknown quantum state. In those schemes the controlled teleportation of an unknown qubit is through a three-qubit GHZ state or W state as the quantum channel along with classical communication.

Entanglement in four-qubit case is more complicated than in three-qubit case. Raussendorf and Briegel [27], showed that cluster states have some particular characters in the case of $N > 3$. For instance, the cluster state is maximally connected and its persistency is better than GHZ-class state. In other words, cluster state has the properties both of the GHZ-class and the W-class entangled states, and is harder to destroy by local operations than GHZ-class states [28, 29].

In this paper, a general scheme for non-maximally entangled controlled teleportation using a four-qubit cluster state is proposed. We prepare a four-particle cluster state for the sender *Alice*, the controller *Charlie* and the receiver *Bob*. The sender *Alice* performs a Bell-basis joint measurement on her qubits in the four-particle cluster state and an unknown qubit. After receiving the measurement result of the sender, the controller *Charlie* takes a joint measurement under a non-maximally entangled Bell-basis. Then receiver *Bob* introduces an auxiliary particle and performs an appropriate unitary transformation on his particles after he cooperates with the controller. Finally *Bob* can reconstruct the unknown state by performing a local measurement on the auxiliary particle and an appropriate unitary transformation on his particle.

2 Controlled Teleportation

We first prepare a cluster state of four particles 1, 2, 3 and 4 [27]

$$|\Psi\rangle_{1234} = \frac{1}{2}\big(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle\big)_{1234} \tag{1}$$

where the particle 1 is kept by sender *Alice*, particles 2 and 3 by the controller *Charlie*, and particle 4 by the receiver *Bob*. Now the sender *Alice* wants to teleport an unknown single-state to *Bob*. This single-state can be usually written as

$$|\Psi\rangle_0 = \alpha|0\rangle_0 + \beta|1\rangle_0, \tag{2}$$

where $|\alpha|^2 + |\beta|^2 = 1$. The state of the composite quantum system is $|\Psi\rangle_{01234} = |\Psi\rangle_0 \otimes |\Psi\rangle_{1234}$.

First, *Alice* performs a Bell-basis joint measurement on her qubits 0 and 1, and the joint state is expressed as

$$|\Psi\rangle_{01234} = \frac{1}{2}\big(|\phi^+\rangle_{01}|\Psi_A\rangle_{234} + |\phi^-\rangle_{01}|\Psi_B\rangle_{234} + |\psi^+\rangle_{01}|\Psi_C\rangle_{234} + |\psi^-\rangle_{01}|\Psi_D\rangle_{234}\big), \tag{3}$$

where the Bell states

$$|\phi^\pm\rangle_{01} = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)_{01}, \qquad |\psi^\pm\rangle_{01} = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)_{01} \tag{4}$$

are four possible measurement results, and

$$
\begin{aligned}
|\Psi_A\rangle_{234} &= \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle - \beta|111\rangle)_{234},\\
|\Psi_B\rangle_{234} &= \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle - \beta|100\rangle + \beta|111\rangle)_{234},\\
|\Psi_C\rangle_{234} &= \frac{1}{\sqrt{2}}(\alpha|100\rangle - \alpha|111\rangle + \beta|000\rangle + \beta|011\rangle)_{234},\\
|\Psi_D\rangle_{234} &= \frac{1}{\sqrt{2}}(\alpha|100\rangle - \alpha|111\rangle - \beta|000\rangle - \beta|011\rangle)_{234},
\end{aligned} \tag{5}
$$

are the corresponding states of qubits 2, 3 and 4 after the measurement by *Alice* with equal probability $1/4$. Then *Alice* tells the result of her measurement to *Charlie* via a classical channel. According to *Alice*'s result, *Charlie* performs appropriate joint unitary transformations on his qubits 2 and 3. For the states of $|\Psi_A\rangle_{234}$, $|\Psi_B\rangle_{234}$, $|\Psi_C\rangle_{234}$ and $|\Psi_D\rangle_{234}$, the corresponding joint unitary transformations are $\mathbf{I}^2 \otimes \mathbf{I}^3$, $\boldsymbol{\sigma}_z^2 \otimes \mathbf{I}^3$, $\boldsymbol{\sigma}_x^2 \otimes \boldsymbol{\sigma}_z^3$ and $i\boldsymbol{\sigma}_y^2 \otimes \boldsymbol{\sigma}_z^3$, respectively. Then the states of qubits 2, 3 and 4 are transformed into the following simple form

$$
|\Psi\rangle_{234} = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle - \beta|111\rangle)_{234}. \tag{6}
$$

To complete the quantum teleportation, the controller *Charlie* needs to perform joint measurement on qubits 2 and 3 under a non-maximally entangled Bell-basis of $\{|e_1\rangle_{23}, |e_2\rangle_{23}, |e_3\rangle_{23}, |e_4\rangle_{23}\}$

$$
\begin{aligned}
|e_1\rangle_{23} &= \cos\theta|00\rangle + \sin\theta|11\rangle, \qquad & |e_2\rangle_{23} &= \sin\theta|00\rangle - \cos\theta|11\rangle,\\
|e_3\rangle_{23} &= \cos\theta|01\rangle + \sin\theta|10\rangle, \qquad & |e_4\rangle_{23} &= \sin\theta|01\rangle - \cos\theta|10\rangle.
\end{aligned} \tag{7}
$$

Then the state in (6) can be written as

$$
|\Psi\rangle_{234} = |e_1\rangle_{23}|\Phi_A\rangle_4 + |e_2\rangle_{23}|\Phi_B\rangle_4 + |e_3\rangle_{23}|\Phi_C\rangle_4 + |e_4\rangle_{23}|\Phi_D\rangle_4 \tag{8}
$$

where

$$
\begin{aligned}
|\Phi_A\rangle_4 &= \frac{1}{\sqrt{2}}(\alpha\cos\theta|0\rangle_4 - \beta\sin\theta|1\rangle_4), \qquad & |\Phi_B\rangle_4 &= \frac{1}{\sqrt{2}}(\alpha\sin\theta|0\rangle_4 + \beta\cos\theta|1\rangle_4),\\
|\Phi_C\rangle_4 &= \frac{1}{\sqrt{2}}(\alpha\cos\theta|1\rangle_4 + \beta\sin\theta|0\rangle_4), \qquad & |\Phi_D\rangle_4 &= \frac{1}{\sqrt{2}}(\alpha\sin\theta|1\rangle_4 - \beta\cos\theta|0\rangle_4).
\end{aligned} \tag{9}
$$

The possible outcomes of *Charlie*'s measurement will be one in $|e_1\rangle_{23}$, $|e_2\rangle_{23}$, $|e_3\rangle_{23}$ and $|e_4\rangle_{23}$ with unequal probabilities, and the state of qubit 4 will correspondingly collapse to $|\Phi_A\rangle_4$, $|\Phi_B\rangle_4$, $|\Phi_C\rangle_4$ and $|\Phi_D\rangle_4$, respectively. Then *Charlie* broadcasts the result including the angle θ of his measurement to *Bob* via a classical channel.

After receiving the *Charlie*'s measurement result, *Bob* introduces an auxiliary qubit in the initial state $|0\rangle_a$. The possible state of the composite system composed of qubit 4 and the auxiliary qubit can be expressed as $|0\rangle_a \otimes |\Phi_A\rangle_4$, $|0\rangle_a \otimes |\Phi_B\rangle_4$, $|0\rangle_a \otimes |\Phi_C\rangle_4$ and $|0\rangle_a \otimes |\Phi_D\rangle_4$. *Bob* needs to performs appropriate unitary transformations on the composite state according to *Charlie*'s result. The appropriate transformation may be either one of the following two kinds of unitary transformations, which depends on the measurement angle θ of *Charlie*. We choose appropriate unitary transformations according to the value of $|\tan\theta|$ being smaller or bigger than 1. It is noticed that such a choice is necessary to guarantee the unitary properties of the following unitary operations.

i) If $|\tan\theta| \leq 1$, *Bob* needs to perform the following unitary transformations U_1 and U_2 under the basis $\{|00\rangle_{a4}, |01\rangle_{a4}, |10\rangle_{a4}$ and $|11\rangle_{a4}\}$

$$\mathrm{U}_1 = \begin{bmatrix} \tan\theta & 0 & \sqrt{1-\tan^2\theta} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{1-\tan^2\theta} & 0 & -\tan\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$\mathrm{U}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \tan\theta & 0 & \sqrt{1-\tan^2\theta} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{1-\tan^2\theta} & 0 & -\tan\theta \end{bmatrix}. \tag{10}$$

Then one gets one of the following four possible states

$$|\Phi_A\rangle_{a4} = \mathrm{U}_1\big(|0\rangle_a|\Phi_A\rangle_4\big) = \frac{1}{\sqrt{2}}\big[\sin\theta|0\rangle_a\big(\alpha|0\rangle_4 - \beta|1\rangle_4\big) + \alpha\cos\theta\sqrt{1-\tan^2\theta}|1\rangle_a|0\rangle_4\big],$$
$$|\Phi_B\rangle_{a4} = \mathrm{U}_2\big(|0\rangle_a|\Phi_B\rangle_4\big) = \frac{1}{\sqrt{2}}\big[\sin\theta|0\rangle_a\big(\alpha|0\rangle_4 + \beta|1\rangle_4\big) + \beta\cos\theta\sqrt{1-\tan^2\theta}|1\rangle_a|1\rangle_4\big],$$
$$|\Phi_C\rangle_{a4} = \mathrm{U}_2\big(|0\rangle_a|\Phi_C\rangle_4\big) = \frac{1}{\sqrt{2}}\big[\sin\theta|0\rangle_a\big(\alpha|1\rangle_4 + \beta|0\rangle_4\big) + \alpha\cos\theta\sqrt{1-\tan^2\theta}|1\rangle_a|1\rangle_4\big],$$
$$|\Phi_D\rangle_{a4} = \mathrm{U}_1\big(|0\rangle_a|\Phi_D\rangle_4\big) = \frac{1}{\sqrt{2}}\big[\sin\theta|0\rangle_a\big(\alpha|1\rangle_4 - \beta|0\rangle_4\big) - \beta\cos\theta\sqrt{1-\tan^2\theta}|1\rangle_a|0\rangle_4\big]. \tag{11}$$

Bob needs to perform measurement on the auxiliary qubit for the composite state (11) under the basis $\{|0\rangle_a, |1\rangle_a\}$. If he finds the state is $|0\rangle_a$, the state of qubit 4 will collapse to

$$|\psi_1\rangle = \alpha|0\rangle_4 - \beta|1\rangle_4, \qquad |\psi_2\rangle = \alpha|0\rangle_4 + \beta|1\rangle_4,$$
$$|\psi_3\rangle = \alpha|1\rangle_4 + \beta|0\rangle_4 \quad \text{or} \quad |\psi_4\rangle = \alpha|1\rangle_4 - \beta|0\rangle_4$$

respectively. For these states, after making corresponding local unitary transformations $\boldsymbol{\sigma}_z^4, \mathbf{I}^4, \boldsymbol{\sigma}_x^4$, or $i\boldsymbol{\sigma}_y^4$, *Bob* reconstructs the teleported state $\alpha|0\rangle_4 + \beta|1\rangle_4$, and the controlled teleportation is realized. Otherwise quantum teleportation fails. Simple calculation yields the probability for a successful controlled teleportation

$$P = 2\sin^2\theta. \tag{12}$$

It is not difficult to find that $0 \leq P \leq 1$, since $|\tan\theta| \leq 1$.

ii) If $|\tan\theta| \geq 1$, *Bob* needs to perform the following unitary transformations U_1' and U_2' under the basis $\{|00\rangle_{a4}, |01\rangle_{a4}, |10\rangle_{a4}$ and $|11\rangle_{a4}\}$

$$\mathrm{U}_1' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cot\theta & 0 & \sqrt{1-\cot^2\theta} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{1-\cot^2\theta} & 0 & -\cot\theta \end{bmatrix},$$
$$\mathrm{U}_2' = \begin{bmatrix} \cot\theta & 0 & \sqrt{1-\cot^2\theta} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{1-\cot^2\theta} & 0 & -\cot\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{13}$$

The corresponding four states are

$$\begin{aligned}
|\Phi_A\rangle_{a4}' &= \mathrm{U}_1'\big(|0\rangle_a|\Phi_A\rangle_4\big) = \frac{1}{\sqrt{2}}\big[\cos\theta|0\rangle_a\big(\alpha|0\rangle_4 - \beta|1\rangle_4\big) - \beta\sin\theta\sqrt{1-\cot^2\theta}|1\rangle_a|1\rangle_4\big], \\
|\Phi_B\rangle_{a4}' &= \mathrm{U}_2'\big(|0\rangle_a|\Phi_B\rangle_4\big) = \frac{1}{\sqrt{2}}\big[\cos\theta|0\rangle_a\big(\alpha|0\rangle_4 + \beta|1\rangle_4\big) + \alpha\sin\theta\sqrt{1-\cot^2\theta}|1\rangle_a|0\rangle_4\big], \\
|\Phi_C\rangle_{a4}' &= \mathrm{U}_2'\big(|0\rangle_a|\Phi_C\rangle_4\big) = \frac{1}{\sqrt{2}}\big[\cos\theta|0\rangle_a\big(\alpha|1\rangle_4 + \beta|0\rangle_4\big) + \beta\sin\theta\sqrt{1-\cot^2\theta}|1\rangle_a|0\rangle_4\big], \\
|\Phi_D\rangle_{a4}' &= \mathrm{U}_1'\big(|0\rangle_a|\Phi_D\rangle_4\big) = \frac{1}{\sqrt{2}}\big[\cos\theta|0\rangle_a\big(\alpha|1\rangle_4 - \beta|0\rangle_4\big) + \alpha\sin\theta\sqrt{1-\cot^2\theta}|1\rangle_a|1\rangle_4\big].
\end{aligned} \tag{14}$$

In this case, one finds for a successful controlled teleportation, the probability is

$$P' = 2\cos^2\theta \tag{15}$$

It is not difficult to find that $0 \leq P' \leq 1$, since $|\tan\theta| \geq 1$.

In both cases, if $|\tan\theta| = 1$, the success probability of controlled teleportation is 100%. This is because, in this case, the measurement basis of controller *Charlie* is the maximally entangled Bell-states.

3 Summary

In this paper, we proposed a new scheme for controlled teleportation using the properties of four-qubit cluster states. In this scheme, a four-particle cluster state is prepared and shared by a sender, a controller and a receiver. The sender first performs a Bell-basis measurement on her qubits, and the controller performs a measurement under a non-maximally entangled Bell-basis after he knows the sender's measurement result. Then the receiver introduces an auxiliary qubit and performs some appropriate unitary transformations on his qubits according to the measure result of the controller. Quantum teleportation is realized after the receiver performs a local measurement on the auxiliary qubit and an appropriate unitary transformation on his qubit. And the probability for a successful teleportation is dependent on the measure angle of the controller.

The difference between the scheme presented here and the other controlled teleportation schemes is that in our scheme the shared state is a four-particle cluster state, and the controller (as the third side) performs a measurement under a non-maximally entangled basis.

Without the cooperation of controller, receiver cannot reconstruct the unknown state by himself. This property of the scheme can be utilized to construct a controlled quantum channel, which may be useful in the future quantum computation.

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